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Constant scalar curvature and warped product globally null manifolds

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Abstract

This paper deals with the curvature properties of a class of globally null manifolds (M, g) which admit a global null vector field and a complete Riemannian hypersurface. Using the warped product technique we study the fundamental problem of finding a warped function such that the degenerate metric g admits a constant scalar curvature on M. Our work has an interplay with the static vacuum solutions of the Einstein equations of general relativity.

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1. Introduction

Semi-Riemannian geometry is the study of smooth manifolds with non-degenerate metric signature (see [11]). The principal special cases are Riemannian geometry, with a positive definite metric, and Lorentz geometry, the mathematical theory used in general relativity (see [3]). Since, for any semi-Riemannian manifold there is a natural existence of null (light-like) subspaces, we refer [8] for a systematic local study on null curves, light-like hypersurfaces and submanifolds of semi-Riemannian manifolds. However, very limited is available on the global geometry of light-like (degenerate) manifolds. With this view in mind, recently, the present author [7] introduced a class of light-like manifolds, called *globally null manifolds* M (see Definition 1) which admit a global null vector field and a complete Riemannian hypersurface. The objective of this paper is to further study on

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globally null manifolds. In Section 2, we brief the basic information needed for the rest of this paper. In Section 3, we show that M can be seen as a warped product of a globally null manifold and a complete Riemannian manifold and conclude that its geometry essentially reduces to the Riemannian geometry of a leaf of its screen distribution. This information is then used in finding the Ricci and the scalar curvatures of M. We study the fundamental problem of finding a warping function on the base manifold of M such that its degenerate warped metric g admits a constant scalar curvature. For this problem we restrict to the study of four-dimensional globally null manifolds and show, by examples, that they have an interplay with some known solutions of the static vacuum Einstein equations and the event horizon or boundary of a black hole in general relativity.

2. Globally null manifolds

Let (M, g) be a real n-dimensional smooth and paracompact manifold where g is a symmetric tensor field of type $(0, 2)$. The radical or the null space of $T_x(M)$, denoted by $\operatorname{Rad} T_x(M)$, is defined by

$$\operatorname{Rad} T_x(M) = \{\xi_x \in T_x(M);\ g(\xi_x, X) = 0,\ X \in T_x(M)\}. \tag{1}$$

The dimension, say r, of $\operatorname{Rad} T_x(M)$ is called nullity degree of g. Rad TM is called the radical distribution of rank r on M. Clearly, g is degenerate or non-degenerate on M iff $r > 0$ or $r = 0$, respectively. We say that (M, g) is a light-like manifold [8] if $0 < r \leq n$. Consider a complementary distribution $S(TM)$ to Rad TM in TM. We call $S(TM)$ a screen distribution on M whose existence is secured for paracompact M. It is easy to see that $S(TM)$ is semi-Riemannian and

$$TM = S(TM) \oplus \operatorname{Rad} TM. \tag{2}$$

Example 1. Consider the unit pseudo sphere S_1^3 of Minkowski space R_1^4 given by the equation $-t^2 + x^2 + y^2 + z^2 = 1$. Cut S_1^3 by the hypersurface $t - x = 0$ and obtain a light-like surface M of S_1^3 with Rad TM spanned by a light-like vector $\xi = \partial_t + \partial_x$. Consider a screen distribution $S(TM)$ spanned by a space-like vector $W = z\partial_y - y\partial_z$. Then, we obtain a light-like surface M, of rank 1, such that TM is spanned by $\{\xi, W\}$.

In this paper, we assume that $r = 1$. Thus, obviously Rad TM is integrable. We need the following results (proofs are available in [7]).

Theorem 1 (Duggal [7]). *Let (M, g) be an n-dimensional light-like manifold with* Rad TM *of rank $r = 1$. Then, there exists a metric* (*Levi-Civita*) *connection ∇ on M with respect to the degenerate metric tensor g.*

Definition 1 (Duggal [7]). A light-like manifold (M, g) is said to be a globally null manifold if it admits a global null vector field and a complete Riemannian hypersurface.

Example 2. Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional globally hyperbolic space–time, with the line element of the metric $\bar{g}$ given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + \bar{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \quad (a, b = 2, \dots, n) \tag{3}$$

with respect to a coordinate system $(t, x^1, \dots, x^n)$ on $\bar{M}$. Choose the range $0 < x^1 < \infty$ so that the metric (3) is non-singular. Take two null coordinates u and v such that $u = t + x^1$ and $v = t - x^1$. Thus, (3) transforms into a non-singular metric: $\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + \bar{g}_{ab}\,\mathrm{d}x^a x^b$. The absence of $\mathrm{d}u^2$ and $\mathrm{d}v^2$ in this transformed metric implies that $\{v = \text{constant}\}$ and $\{u = \text{constant}\}$ are light-like hypersurfaces of M. Let $(M, g, v = \text{constant})$ be one of this light-like pair and let D be the one-dimensional distribution generated by the null vector $\{\partial_v\}$, in $\bar{M}$. Denote by L the one-dimensional integral manifold of D. A leaf M' of the $(n-1)$-dimensional screen distribution of M is Riemannian with metric $\mathrm{d}\Omega^2 = \bar{g}_{ab} x^a x^b$ and is the intersection of the two light-like hypersurfaces. In particular, there will be many global time-like vector fields in globally hyperbolic space–time $\bar{M}$. If one is given a fixed global *time function* then its gradient is a global time-like vector field in a given $\bar{M}$. With this choice of a global time-like vector field in $\bar{M}$, we conclude that both its light-like hypersurfaces admit a global null vector field. Now, the celebrated Hopf–Rinow theorem allows to assume that M' is a complete Riemannian hypersurface of M. Thus, there exists a pair of globally null hypersurfaces of a globally hyperbolic space–time. In particular, a Minkowski space and a De-Sitter space $\bar{M}$ have a pair of globally null hypersurfaces. Proceeding similar to above example for four-dimensional $\bar{M}$, one can show that Robertson–Walker, Reissner–Nordström and Kerr space–times have pairs of globally null hypersurfaces.

Using Theorem 1, we construct an $(n+1)$-dimensional Lorentz manifold $(\bar{M}, \bar{g})$ with coordinates $(x^1; x^a, y)$, where $(x^1; x^a, y = \text{constant})$ are coordinates on M induced by the foliation determined by Rad TM and (y) is a coordinate on one-dimensional fibre of its vector bundle structure. Thus, g is degenerate metric on a family of globally null manifolds (M, g), induced by the Lorentz metric $\bar{g}$ of $\bar{M}$. We use this structure to find a suitable Frenet frame for M, along a null curve C in an n-dimensional globally null manifold (M, g), with $n > 1$, given by

$$C : (x^1(t), x^2(t), \dots, x^n(t), y(t) = a), \quad t \in I \subset R \tag{4}$$

for a coordinate neighbourhood $\mathcal{U}$ on C. Then, the tangent vector field

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\mathrm{d}x^1}{\mathrm{d}t}, \dots, \frac{\mathrm{d}x^n}{\mathrm{d}t}, 0\right)$$

on $\mathcal{U}$ satisfies $g(\mathrm{d}/\mathrm{d}t, \mathrm{d}/\mathrm{d}t) = 0$, where the scalar products are with respect to the signature $(- + \dots +)$ of $\bar{g}$. Denote by TC the tangent bundle of C and consider a class of null curves such that Rad $TM = TC$ and both are generated by the null vector field ξ. Let $S(TM)$ be a complementary screen distribution to Rad TM. Then, (2) reduces to

$$TM = \text{Rad}\,TM \oplus S(TM) = TC \oplus S(TM). \tag{5}$$

Set $\mathrm{d}/\mathrm{d}t = \xi$. Since, by Theorem 1, ∇ is a metric connection on M (i.e., $\nabla g = 0$), using this and (5), we obtain the following equations:

$$\begin{aligned}
\nabla_\xi \xi &= h\xi,\\
\nabla_\xi W_1 &= -k_1\xi + k_3 W_2 + k_4 W_3,\\
\nabla_\xi W_2 &= -k_2\xi - k_3 W_1 + k_5 W_3 + k_6 W_4,\\
\nabla\xi W_3 &= -k_4 W_1 - k_5 W_2 + k_7 W_4 + k_8 W_5,\\
&\ \vdots\\
\nabla_\xi W_{n-2} &= -k_{n-1} W_{n-4} - k_n W_{n-3} + k_{n+2} W_{n-1} + k_{n+3} W_n,\\
\nabla_\xi W_{n-1} &= -k_{2n-4} W_{n-3} - k_{2n-3} W_{n-2},
\end{aligned} \tag{6}$$

provided $n \geq 5$, where h and $\{k_1, \ldots, k_{2n-3}\}$ are smooth functions on $\mathcal{U}$ and $\{W_1, \ldots, W_{n-1}\}$ is an orthonormal basis of $\Gamma(S(TM)_{\mathcal{U}})$. See [7] for three special cases when $1 < n < 5$. In general, for any $n > 1$, we call

$$F = \left\{\frac{\mathrm{d}}{\mathrm{d}t} = \xi, W_1, \ldots, W_{n-1}\right\}, \tag{7}$$

a Frenet frame on M along C with respect to the screen distribution $S(TM)$. The functions $\{k_1, \ldots, k_{2n-3}\}$ and Eqs. (6) are called curvature functions of C and Frenet equations for F, respectively.

Remark 1. Since the Frenet frame (7) only involves the base vector fields on M and not of any landing Lorentz manifold, it is a suitable frame to study global properties of M, which is our objective.

The following result was proved in [7] for any light-like manifold, with Rad TM of rank 1, which also holds for a globally null manifold.

Proposition 1. *Let C be a null curve of a globally null manifold* (M, g), *defined by Eq.* (4). *Then, there exists a parameter p with respect to which C is null geodesic, generated by a global null vector field.*

Example 3. Consider a four-dimensional Minkowski space–time R^4_1, with a Lorentz metric of signature $(-+++)$. As explained in Example 2, let (M, g) be one of the pair of globally null hypersurfaces of R^4_1. Consider a curve C in M defined by $C : (p, -p, a_1, a_2)$, $p \in I \subset R$, where a_1 and a_2 are suitable constants. Then, $\mathrm{d}/\mathrm{d}p = (1, -1, 0, 0)$ is a null tangent vector field, say ξ, of a null curve C of M. Moreover, $\nabla_\xi \xi = 0$. Therefore, C is a null geodesic of M. Choose a Frenet frame $\{\xi, W_1, W_2\}$ on M along C where

$$W_1 = (bp, -bp, 1, 0), \qquad W_2 = (cp, -cp, 0, 1)$$

are orthonormal space-like vectors which generate a screen distribution of M; b and c are suitable constants. Following are the rest of two Frenet equations:

$$\nabla_\xi W_1 = b\xi + 0W_2, \qquad \nabla_\xi W_2 = c\xi + 0W_1,$$

such that, for the special case $n = 3$, $k_1 = -b$, $k_2 = -c$, $k_3 = 0$.

In general, let (M, g) be a globally null hypersurface of a class of globally hyperbolic space–times (such as Schwarzchild, Robertson–Walker, Reissner–Nordström and Kerr space–times). It is known [8] that, for this class of hypersurfaces, their respective screen distribution is topologically a 2-sphere S^2. It is easy to show that a null geodesic curve of any one of such hypersurfaces M, projected to its screen distribution, is a great circle of S^2.

3. Warped product

In 1969, Bishop and O'Neill [5] introduced a class of warped product manifolds as follows. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, $f : M_1 \to (0, \infty)$ and $\pi : M_1 \times M_2 \to M_1$, $\eta : M_1 \times M_2 \to M_2$ the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in M_1 \times M_2$. Denote the warped product manifold $\bar{M} = (M_1 \times_f M_2, \bar{g})$, where

$$\bar{g}(X, Y) = g_1(\pi_\star X, \pi_\star Y) + (f \circ \pi)^2 g_2(\eta_\star X, \eta_\star Y) \tag{8}$$

for every X and Y of $\bar{M}$, and $\star$ is symbol for the tangent map. They proved that $\bar{M}$ is a complete Riemannian manifold iff both M_1 and M_2 are complete Riemannian manifolds. In particular, they constructed a large variety of complete Riemannian manifolds of everywhere negative sectional curvature using warped products. In 1980, Beem–Ehrlich [3] proved that the Lorentzian warped product $\bar{M}$ is globally hyperbolic if and only if M_1 is globally hyperbolic and M_2 is a complete Riemannian manifold. Using warped product, they did an extensive global study on causal and completeness properties of Lorentz manifolds. Motivated by very effective use of warped products in the global study of Riemannian and Lorentzian geometry, we introduce a new class of *products in light-like geometry* as follows.

Let (N, g_N) and (F, g_F) be a light-like and a Riemannian manifold of dimensions n and m, respectively, where the Rad TN is of *rank* r. Let $\pi : N \times F \to N$ and $\eta : N \times F \to F$ denote the projection maps given by $\pi(x, q) = x$ and $\eta(x, q) = q$ for $(x, q) \in N \times F$, respectively, where the projection π on N is with respect to the non-degenerate screen distribution $S(TN)$.

Definition 2. The product manifold $M = N \times F$ is said to be a light-like warped product $N \times_f F$, with the degenerate metric g defined by

$$g(X, Y) = g_N(\pi_\star X, \pi_\star Y) + (f \circ \pi)^2 g_F(\eta_\star X, \eta_\star Y) \tag{9}$$

for every X, Y of M, and $\star$ is the symbol for the tangent map.

It follows that Rad TM of M still has *rank* r but $\dim(M) = n + m$ and $\dim(S(TM)) = n + m - r$. In particular, if N is a globally null manifold and F is a complete Riemannian manifold, then we have the following result (the proof is similar to the proof of Theorem 3 in [7]).

Theorem 2. *Let $M = (N \times_f F, g)$ be a light-like warped product manifold of a globally null manifold (N, g_N) and a complete Riemannian manifold (F, g_F) of dimensions n and m, respectively. Then, the following assertions are equivalent*

(a) *The screen distribution $S(TM)$ is integrable.*
(b) $M = L \times M'$ *is a global null product manifold, where* L *is* a one-*dimensional integral manifold of the global null curve* C *in* M *and* (M', g') *is a complete Riemannian hypersurface of* M *which is a triple warped product*

$$M = L \times B \times_f F, \qquad M' = (B \times_f F, g'), \tag{10}$$

where (B, g_B) *is a complete Riemannian hypersurface of* $N = L \times B$.
(c) $S(TM)$ *is parallel with respect to the metric connection on* M.

Example 4. Let R_1^{d+1} be a Lorentz space with the metric $\tilde{g}$ given by

$$\tilde{g}(x, y) = -x^0 y^0 + \sum_{i=1}^{d} x^i y^i$$

with respect to a coordinate system $(x^o, \dots, x^d)$. Consider a smooth function $f : D \to R$, where D is an open set of R^d. Then

$$M = \{(x^0, \dots, x^n \in R_1^{d+1};\ \ x^0 = f(x^1, \dots, x^d)\}, \tag{11}$$

is a hypersurface of R_1^{d+1} which is called a *Monge hypersurface*. Let natural parameterization on M be given by

$$x^0 = f(v^0, \dots, v^{d-1}), \qquad x^{\alpha+1} = v^\alpha, \quad \alpha \in \{0, \dots, n-1\}.$$

Hence, the natural frames field on M is globally defined by

$$\partial_{v^\alpha} = f'_{x^{\alpha+1}} \partial_{x^0} + \partial_{x^{\alpha+1}}, \quad \alpha \in \{0, \dots, d-1\}.$$

Then, it follows that $TM^\perp$ is spanned by a global vector

$$\xi = \partial_{x^0} + \sum_{i+1}^{d} f'_{x^i} \partial_{x^i}. \tag{12}$$

We know from (1) that M is light-like if and only if $TM^\perp = \operatorname{Rad} TM$. This means that ξ, defined by (12), is a null vector field. Hence, we may state

Proposition 2. *A Monge hypersurface M, given by (11), is light-like if and only if the function f is a solution of the differential equation* $\sum_{i=1}^{d} (f'_{x^i})^2 = 1$.

Proposition 3. *The screen distribution of a light-like Monge hypersurface M of* R_1^{d+1}*, given by (11), is integrable.*

Proof. Choose a null transversal vector field (which is not tangent to M) given by $V = (1/2)\{-\partial_{x^0} + \sum_{i=1}^{d} f'_{x^i} \partial_{x^i}\}$. It follows that $g(\xi, V) = 1$. Let $\bar{\nabla}$ be the Levi-Civita connection, with respect to the Minkowski metric $\bar{g}$, on R_1^{d+1}. Then, for any two vectors $X, Y \in \Gamma(S(TM))$, the Lie bracket $[X, Y] \in \Gamma(S(TM))$. Indeed

$$\bar{g}([X, Y], V) = \bar{g}(\bar{\nabla}_X Y - \bar{\nabla}_Y X, \partial_{x^0}) = -\{\bar{g}(X, \bar{\nabla}_Y \partial_{x^0}) - \bar{g}(Y, \bar{\nabla}_X \partial_{x^0})\} = 0.$$

Hence, $S(TM)$ is integrable which completes the proof. □

To relate above example with this section, we let R_1^{d+1} admit a globally defined time-like vector field, which means it is a time orientable Minkowski space–time. This further means that R_1^{d+1} is a globally hyperbolic space–time. By Proposition 3, with the equivalent assertions (a) and (b) of Theorem 2, we conclude that $M = L \times M'$ is a *globally null Monge hypersurface* of a Minkowski space–time R_1^{d+1}, where M' is a complete Riemannian manifold. Furthermore, M can be seen as a triple warped product manifold, as explained above, if we set $d = n + m$.

The Ricci tensor of M is $\mathrm{Ric}(X, Y) = \mathrm{trace}\{Z \to R(X, Z)Y\}$, for vector fields X, Y, Z of M, where $R(X, Z)Y$ is the curvature tensor of M. With respect to a natural frames field $\{\xi = \partial_{x^0}, \partial_{x^1}, \dots, \partial_{x^{n-1}}\}$, we have

$$\begin{aligned} \mathrm{Ric}(X, Y) &= g^{ab} g(R(\partial_{x^a}, X)\partial_{x^b}, Y) \quad (a, b = 1, \dots, n-1) \\ &= \mathrm{Ric}'(X', Y'), \end{aligned} \tag{13}$$

where Ric' denotes the Ricci tensor of M' and $X', Y' \in M'$. In terms of a quasi-orthonormal Frenet frame $\{\xi, W_1, \dots, W_{n-1}\}$ along a null curve C (see Eq. (7)), the degenerate metric g and (13) are expressed by

$$g(X, Y) = \sum_{a=1}^{n-1} g(X, W_a) g(Y, W_a),$$

$$\mathrm{Ric}(X, Y) = \sum_{a=1}^{n-1} g(R(W_a, X)W_a, Y) = \mathrm{Ric}'(X', Y'). \tag{14}$$

Using (13) and Theorem 2, we have the following important result.

Proposition 4. *Let (M, g) be a globally null manifold, with M' its complete Riemannian hypersurface. Then, the metric g and the Ricci tensor of M can be determined from a set of data specified entirely on M'. Also, M and M' have same Ricci tensors. Moreover, if M is a globally null warped product manifold, then the geometry of M reduces to the Riemannian geometry of its warped product hypersurface $M' = (B \times_f F, g')$, as defined by* (10).

For example, let M be a globally null Monge hypersurface, defined by (11), of a Minkowski space–time R_1^{d+1}. Following two results hold (for proof see [8] where R_1^4 case is discussed since the general case is very lengthy).

(a) Let M' be any leaf of the screen distribution of M and R, R' the curvature tensors of M and M', respectively. Then, $R = (1/2)R'$.
(b) M is (1) flat, (2) totally geodesic, (3) totally umbilical, (4) minimal if and only if M' is so immersed as a submanifold of R_1^{d+1}.

In other words, M' is an invariant hypersurface of M. This means that any tensor (including the degenerate metric g) or geometric object on M can be determined from a set of data specified entirely on its Riemannian hypersurface M'. Thus, one can essentially do all the analysis on the complete Riemannian hypersurface M' of M. Moreover, we have

(1) The null leaves $N \times q$, $q \in F$, of warped product M, can be induced to the space-like leaves $B \times q$, and are totally geodesic in M.

(2) The fibres $(p_0, p) \times F$, $p_0 \in L$ and $p \in B$ can be induced to the space-like fibres $p \times F$, and are totally umbilical in M.
(3) For each $(p, q) \in M'$, the induced leaf $B \times q$ and the induced fibre $p \times F$ are orthogonal at (p, q).
(4) The gradient of the lift $h \circ \pi$ of a smooth function $h \in N$ is the lift to M of the gradient of h on its induced Riemannian manifold B.

In general, for a covariant tensor $T \in N$, its lift $\bar{T} \in M$ is the pullback $\pi^\star(T)$ under the map $\pi : M \to B \subset N$. This is why, even if the metric g_N of N is degenerate, all tensors and geometric objects and their pullback are with respect to the induced Riemannian metric g_B of B. The vectors tangent to leaves and fibres are called *horizontal* and *vertical*, respectively. The lift to M of the Hessian of a smooth function f on N, denoted by H^f, agrees with the Hessian of the lift $f \circ \pi$ on the horizontal vectors of B. We denote Ric^B for the pullback by π of Ric' and similarly for Ric^F.

Proposition 5. *Let $M = L \times B \times_f F$ be an $(n+m)$-dimensional triple warped product manifold with $\dim(F) = m > 1$. Then*

(1) $\mathrm{Ric}(\xi, \xi) = \mathrm{Ric}(\xi, X) = \mathrm{Ric}(X, U) = 0$, $\xi \in L$,
(2) $\mathrm{Ric}(X, Y) = \mathrm{Ric}'(X, Y) = \mathrm{Ric}^B(X, Y) - (m/f)H^f(X, Y)$,
(3) $\mathrm{Ric}(U, V) = \mathrm{Ric}^F(U, V) - \langle U, V\rangle\{\Delta f/f + (m-1)\langle\nabla f, \nabla f\rangle/f^2\}$,

where $\Delta f = \mathrm{trace}(H^f)$ is the Laplacian of f, $\nabla f = \mathrm{grad}(f)$, X, Y horizontal and U,V vertical vector fields.

Proof. Use Proposition 4 and follow Corollary 7.43 in [11]. □

We use the following identifications of $T_x(M)$ for any $x = (p_0, p, q) \in M$.

$$T_x(L \times B \times_f F) \cong T_x(L \times B \times F) \cong T_{p_0}(L) \times T_p(B) \times T_q(F),$$

$$T_x(L \times B \times_f F) \xrightarrow{\text{projected}} T_{(p,q)}(B \times_f F) \cong T_{(p,q)}(B \times F)$$

A Frenet frame $\{\xi, W_1, \ldots, W_{n-1}\}$ on $T_{(p_0,p)}N$ (see Eq. (7)) is identified to an orthonormal basis $\{W_a\}$ $(a = 1, \ldots, n-1)$ on T_pB. Any horizontal vector $X_{(p_0,p,q)} \in M$ is identified to a horizontal vector $\bar{X}_{p,q} = (X_p, 0_q) \in B$. Similar notations follow for vertical vectors and tensors. We denote S^B the pullback by π of the scalar curvature of B and similar for S^F. For the degenerate metric g, at a point $x = (p_0, p, q) \in M$, we have

$$g_x \xrightarrow{\text{projected}} g'_{(p,q)} = (g_p, g_q), \qquad \bar{g}_{B(p,q)} = (g_p, 0_q), \qquad g_{F(p,q)} = (0_p, g_q),$$

where g', g_B and g_F are Riemannian metrics on M', B and F, respectively.

Proposition 6. *Suppose S is the scalar curvature of a triple warped product manifold $M = L \times B \times_f F$, with $\dim(F) = m > 1$. Then*

$$S = S' = S^B + \frac{S^F}{f^2} - 2m\frac{\Delta f}{f} - m(m-1)\frac{\langle\nabla f, \nabla f\rangle}{f^2}, \tag{15}$$

where S' is the induced scalar curvature of $M' = (B \times_f F, g')$.

Proof. Let $\{\xi; W_a\}$ be a pseudo-orthonormal basis for $T_{(p_0,q)}(L \times B)$ so that $\{W_a\}$ is an orthonormal basis for $T_p B$. Then, by isomorphism, $\{\bar{W}_a = (W_a, 0_q)\}$ is an orthonormal set in $T_{(p,q)}(B \times_f F)$. Choose a set $\{W_i\}$ of m vectors on $T_q F$ such that $\{\bar{\xi}; \bar{W}_a; \bar{W}_i\}$ forms a pseudo-orthonormal basis for $T_{(p_0,p,q)} M$. Thus, $\{\bar{W}_a; \bar{W}_i\}$ is an orthonormal basis for $T_{(p,q)}$. Since

$$g_F(\bar{W}_i, \bar{W}_i) = f^2(W_i, W_i) = g_F(fW_i, fW_i) = 1,$$

we conclude that $\{fW_i\}$ is an orthonormal basis for $T_q F$. Using Proposition 3, for each a and each i, we get

$$\mathrm{Ric}(\bar{W}_a, \bar{W}_a) = \mathrm{Ric}'(\bar{W}_a, \bar{W}_a) = \mathrm{Ric}^B(\bar{W}_a, \bar{W}_a) - \frac{m}{f} H^f(\bar{W}_a, \bar{W}_a),$$

$$\mathrm{Ric}(\bar{W}_i, \bar{W}_i) = \mathrm{Ric}^F(\bar{W}_i, \bar{W}_i) - f\left[\Delta f + (m-1)\frac{\langle \nabla f, \nabla f\rangle}{f}\right]$$

Hence, using $g(\xi, \xi) = 0$, $g(W_a, W_a) = g(W_i, W_i) = 1$, we get

$$\begin{aligned} S(p_0, p, q) \xrightarrow{\text{projected}} S'(p, q) &= R_{\alpha\alpha} \quad (2 \le \alpha \le n+m) \\ &= \mathrm{Ric}(\bar{W}_a, \bar{W}_a) + \mathrm{Ric}(\bar{W}_i, \bar{W}_i) \\ &= S^B + \frac{S^F}{f^2} - 2m\frac{\Delta f}{f} - m(m-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2}. \qquad \square \end{aligned}$$

4. Constant scalar curvature

In this section, we deal with the following fundamental problem.

Given a fibre F, of M, with constant scalar curvature, find a warping function f on its base manifold $L \times B$ such that the degenerate warping metric g admits a constant scalar curvature on $M = (L \times B \times_f F, g)$.

We restrict to $\dim(M) = n = 4$ since this case has an interplay with some known exact solutions of the static vacuum Einstein equations and the event horizon or boundary of a black hole in general relativity. For this case, either (1) $\dim(B) = 1$ and $\dim(F) = 2$ or (2) $\dim(B) = 2$ and $\dim(F) = 1$. We deal with both these subcases separately. Using the material of previous two sections, we first work on the Riemannian warped product manifold (M', g') and then show how to glue g' with the degenerate metric g of M.

Case 1. $\dim(B) = 1$ and $\dim(F) = 2$.

Theorem 3. *Let $M = (L \times B \times_f F, g)$ be a* four-*dimensional globally null warped product manifold, $B = (a, b)$ an open connected subset of real line with positive definite metric* $\mathrm{d}r^2$ *and $-\infty \le a < b \le +\infty$ and the fibre space F be of constant scalar curvature $c \neq 0$. Then, g admits the following warping functions $f(r)$ for which M has a constant scalar curvature k.*

(i) $k > 0, \quad f(r) = \sqrt{\frac{3c}{k}}\left[\tan^2\left(\pm\left(\frac{k}{6}\right)^{1/2} r + c_1\right) + 1\right]^{-1/2}, \quad c > 0,$

(ii) $k=0,\quad f(r)=\pm\left(\sqrt{\frac{c}{2}}\right)r+c_1,\quad c>0,$

(iii) $k<0,\quad f(r)=c_1\exp\left(\sqrt{\frac{-k}{6}}r\right)-\frac{3c}{4c_1k}\exp\left(-\sqrt{\frac{-k}{6}}r\right),$

where c_1 is a constant such that $f(r)$ is real and positive.

Proof. Let $f(r)=u^{2/3}$. Then, $\Delta f=f''$ and $\langle\nabla f,\nabla f\rangle=(f')^2$. Using this with $S^B=0$, $S^F=c$ and $S=k$ in (15), we obtain

$$u''+\tfrac{3}{8}ku-\tfrac{3}{8}cu^{-1/3}=0. \tag{16}$$

Let $u'=y$ so that $\mathrm{d}y/\mathrm{d}r=u''$. Using this in (16), separating variables and then integrating both sides, we obtain

$$y=\pm(3/8)^{1/2}u\sqrt{3cu^{-4/3}-k}=\frac{\mathrm{d}u}{\mathrm{d}r}.$$

Therefore:

$$\frac{\mathrm{d}u}{u\sqrt{3cu^{-4/3}-k}}=\pm\left(\frac{3}{8}\right)^{1/2}\mathrm{d}r.$$

Following are three cases of the integral of above equation:

$$k>0,\quad v^2=k\tan^2\left(\pm\left(\frac{k}{6}\right)^{1/2}r+c_1\right),\qquad k=0,\quad f(r)=\pm\left(\sqrt{\frac{c}{2}}\right)r+c_1,$$

$$k<0,\quad \ln\left|\frac{v-\sqrt{-k}}{v+\sqrt{-k}}\right|=\pm\left(-\frac{k}{6}\right)^{1/2}r,$$

where we set $v^2=3cu^{-4/3}-k$. From above three equations the results of this theorem follow easily for the case of Riemannian warped product manifold (M',g'). To complete the proof, we now show how to glue g' with the degenerate metric g of M. It follows from the Proposition 4 that the scalar curvatures of M' and M are same. The warping function $f_p\in B$ can be glued with the warping function $f_{(p_0,p)}=(0_{p_0},f_p)\in L\times B$. Based on information from Section 2, the Riemannian metric g' can be glued with the degenerate metric g, of M, as follows:

$$g=\begin{pmatrix}O_{1,1} & O_{1,3}\\ O_{3,1} & g'\end{pmatrix},$$

where $g'(X,Y)=g_B(\pi_\star X,\pi_\star Y)+(f\circ\pi)^2g_F(\eta_\star X,\eta_\star Y)$. □

Corollary. *If $c=0$, then following are the warping functions $f(r)$ for which M has a constant scalar curvature k.*

(i) $k>0,\quad f(r)=\left[c_1\cos\left(\sqrt{\frac{3k}{8}}r\right)+c_2\sin\left(\sqrt{\frac{3k}{8}}r\right)\right]^{2/3},$

(ii) $k = 0, \quad f(r) = (c_1 r + c_2)^{2/3}$,

(iii) $k < 0, \quad f(r) = \left[c_1 \exp\left(\sqrt{\frac{-3k}{8}}r\right) + c_2 \exp\left(-\sqrt{\frac{-3k}{8}}r\right)\right]^{2/3}$,

where c_1 and c_2 are constants such that $f(r)$ is real and positive.

Physical model 1. Let $M = (L \times M', g)$ be a four-dimensional globally null manifold, with (M', g') its complete space-like hypersurface. Also, let $(\tilde{M}, \tilde{g})$ be a four-dimensional globally hyperbolic space–time manifold of general relativity. By definition, $\tilde{M}$ has a complete space-like hypersurface H (called *Cauchy surface*) such that $\tilde{M} = R \times H$. In the following we show, by means of a physical example, that H is a warped product manifold of Case 1, and the set $\{M, M', \tilde{M}\}$ of these three manifolds has the following interplay.

$$(M, g) \supset (M' = B \times_f F, g') \subset (\tilde{M} = R \times H, \tilde{g}), \tag{17}$$

where $(H = B \times_{\tilde{f}} F, g_H)$ and $\tilde{f}$ is a warping function on $R \times B$.

Example 5. Let $(\tilde{M}, \tilde{g})$ be the Schwarzschild space–time with the metric

$$\tilde{g} = -A(r)\,\mathrm{d}t^2 + A^{-1}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{s^2}, \quad A(r) = 1 - 2mr^{-1} > 0,$$

where S^2 is totally geodesic 2-sphere of radius $2m$ and m is positive mass. This metric represents the most important non-trivial solution of the static vacuum Einstein field equations. It is well known that $\tilde{M}$ is a globally hyperbolic manifold [3]. To relate this with Eq. (17), we consider the following conformal deformation metric $\bar{g}_H$ defined by

$$\bar{g}_H = A(r) g_H = \mathrm{d}r^2 + A(r) r^2\,\mathrm{d}\Omega^2_{s^2}.$$

If we set B a one-dimensional space with metric $\mathrm{d}r^2$ and S^2 a two-dimensional fiber space F of $\tilde{M}$, then using (17) we conclude that the Schwartzchild space–time $\tilde{M}$ has a complete Cauchy hypersurface

$$(\bar{H} = B \times_{\bar{f}} F, \bar{g}_H), \quad \bar{f} = \tilde{f}\sqrt{A(r)},$$

and it has an interplay with a four-dimensional globally null manifold M. It is well known that static solutions of space–times are closely connected with an open Riemannian 3-manifold containing a 2-sphere, occurring at the event horizon or the boundary of a black hole in general relativity. This physical relation is apparent in above example and, more generally, in many solutions of the static vacuum equations of asymptotically flat space–times, which have 2-spheres near infinity. Relevant to this paper, we have further demonstrated, through Eq. (17), that 2-sphere can act as a common link between the three manifolds M, M' and $\tilde{M}$, relating the geometries of globally null and the globally hyperbolic manifolds. Finally, to relate this example with Theorem 3, consider the case $k = 0$ (others are similar) so that

$$f(r) = \pm\left(\sqrt{\frac{c}{2}}\right) r + c_1.$$

In this example, $F = S^2$ whose scalar curvature is $1/4m^2$ and

$$\bar{f} = r\sqrt{1 - 2mr^{-1}}.$$

Matching $f = \bar{f}$, and $c = 1/4m^2$, we obtain

$$(1 - 8m^2)r^2 + (16m^3 \pm 4\sqrt{2}mc_1)r + 8mc_1^2 = 0,$$

where c_1 is such that r has real solutions from above equation.

Remark 2. For similar study on higher dimensional Riemannian and semi-Riemannian warped product manifolds, we suggest the works of Dobarro–Dozo [6] and Ehrlich et al. [9], respectively.

Case 2. $\dim(B) = 2$ and $\dim(F) = 1$.

Here we follow Yamabe [12] for the existence of constant curvature metrics on a three-dimensional compact Riemannian manifold $(M' = B \times_f F, g')$, with $\dim(B) = 2$, and, then follow Anderson's [1] recent work on their relation with the static vacuum Einstein equations. We study two specific subcases and glue the degenerate metric g of $(M = L \times M', g)$ with g'. Denote by $\mathcal{M}$ the space of all smooth Riemannian metrics on M' and $\mathcal{M}_1 \subset \mathcal{M}$ of metrics satisfying $\mathrm{vol}_{g'} M' = 1$. Define the total scalar curvature or Einstein–Hilbert action $\mathcal{S} : M' \to R$ by

$$\mathcal{S}(g') = v^{-1/3} \int_{M'} S^{M'} \,\mathrm{d}V_{g'}, \tag{18}$$

where $S^{M'}$ is the scalar curvature of M', $\mathrm{d}V_{g'}$ the volume element and v the volume of M'. The critical points of $\mathcal{S}$ are *Einstein metrics*. Moreover, only in dimension 3 these Einstein metrics are of constant scalar curvature. There is a well known procedure to obtain Einstein manifolds. Following Yamabe [12], suppose $[g']$ is a conformal class of any metric $g' \in \mathcal{M}_1$. Then there exists a metric $\bar{g} \in \mathcal{M}_1$ which achieves its infimum $\mu[g'] \equiv \mathcal{S}|_{[g'] \cap \mathcal{M}_1}$. Such metrics are called *Yamabe metrics*. However, there are restrictions on the existence of Yamabe metrics. Denote by $\sigma(M') = \sup(\mu[g'])_{\mathcal{C}_1}$, where $\mathcal{C}_1$ is the subset of unit volume Yamabe metrics. If $\sigma(M') \leq 0$, it has been proved by Besse [4] that any Yamabe metric $g_0 \in \mathcal{C}_1$ such that $S^{M'}_{g_0} = \sigma(M')$ is Einstein. Otherwise, this problem still remains open. Under these restrictions, it is reasonable to say that there exists a four-dimensional globally null manifold M whose three-dimensional compact Riemannian hypersurface (M', g') is an Einstein manifold with a constant curvature k and g' is a Yamabe metric. Then, it follows from Proposition 4 that M is also of constant scalar curvature k. Now we show, that for two specific subcases, of Case 2, there exists a warping function f on the base manifold B of $(M' = B \times_f F, g')$, such that the metric g has a constant scalar curvature on M.

Subcase (i). $M' = S^3$ and $F = S^1$.

Let $\{g'_i\}$ be a maximum sequence of unit volume Yamabe metrics on M'. It has been shown by Anderson [1] that the degenerations of such a sequence are described by solutions to the static vacuum Einstein equations. Consider a specific class of such solutions, known as Weyl solutions, where (M', g') is a warped product of Case 2 and

$$M' = B \times_f S^1, \qquad g' = g_B + f^2 \,\mathrm{d}\theta^2, \tag{19}$$

where (B, g_B) is a Riemannian surface (see [2] for details) and f is a positive function on B. Let there be symmetry on B such that $f = f(r)$ with respect to a coordinate system (r, ϕ) at any point $p \in B$. This is possible, in particular, if we set $M' = S^3$ so that B is a two-dimensional solid torus. Then, S^3 can be seen as the union of two solid tori $B \times S^1$, glued along the torus boundary $\partial(B)$ by interchanging the two circles in $\partial(B)$. Assume that outside a compact set, g' is isometric to a rank 2 hyperbolic cusp, so that in particular, $f = f(r) = \mathrm{e}^{-r}$ for large r of S^2. For the existence of such a metric g' see Theorem 4.32 in [4]. Anderson [1] has shown that such a warped product metric g' is invariant under the S^1 action on $B \times S^1$ and the scalar curvature of M' is -6. Therefore, based of Proposition 2, the scalar curvature of the globally null manifold M is also -6 and its degenerate metric g can be glued with the Riemannian metric g' as explained in the proof of Theorem 4. Furthermore, there are conditions (discussed in [1]), under which the degeneration corresponds to non-trivial vacuum solutions of the Einstein equations of four-dimensional space–time manifolds of general relativity. Examples are several asymptotically flat space–times (including the Schwarzchild space–time) which have 2-spheres near infinity. Thus, we have the following physical model for Case 2.

Physical model 2. Let $M = (L \times M', g)$ be a four-dimensional globally null manifold, with (M', g') a warped product of Case 2 and of the form (19), such that $M' = S^3$ and B is a solid torus. Also, let $(\tilde{M}, \tilde{g})$ be a class of asymptotically flat space–times which have 2-spheres near infinity. Then, the set of these three manifolds $\{M, M', \tilde{M}\}$ has the following interplay

$$(M, g) \supset (S^3 \times_f S^1, g') \subset (\tilde{M}, \tilde{g}),$$

where $\tilde{M}$ can have a suitable warped product structure.

Subcase (ii). $S^F = 0$ on F.

Theorem 4. *Let* $M = (L \times B \times_f F, g)$ *be a* four-*dimensional globally null warped product manifold,* (B, g_B) *a Riemannian surface with scalar curvature* S^B *and* $F = (a, b)$ *an open connected subset of real line with positive definite metric* $\mathrm{d}x^2$ *and* $-\infty \leq a < b \leq +\infty$. *Then, the metric g admits infinitely many warped functions for which M has constant scalar curvature.*

Proof. Following Ehrlich et al. [9], we let $H_{1,2}(B)$ denote the Sobolev space of functions on B whose first order derivatives are in the norm space $L_2(B)$ and L is the differential operator on $H(B)$ such that $L(f) = -\Delta f + (1/2)S^B f$. Consider the first eigenvalue λ on L given by

$$\lambda = \min_{f \neq 0 \in H(B)} \frac{\int_B f L(f)\, \mathrm{d}V}{\int_B f^2\, \mathrm{d}V} = \min_{f \neq 0 \in H(B)} \frac{\int_B |\Delta f|^2\, \mathrm{d}V + (1/2) \int_B S^B f^2\, \mathrm{d}V}{\int_B f^2\, \mathrm{d}V}.$$

Set $\lambda = (1/2)\bar{k}$. Then

$$L(f) = \lambda f = \tfrac{1}{2}\bar{k} f \qquad (20)$$

implies that f is a eigenvalue function of the operator L. Kazdan–Warner [10] have shown that such an eigenfunction is never zero, positive and smooth. Thus, we assume that $f > 0$ on B. Finally, it follows from Eq. (15), with $\dim(F) = m = 1$, $S^F = c = 0$

and above that

$$\Delta f + \tfrac{1}{2}(\bar{k} - S^B) f = 0.$$

Therefore, there exists a warped function f such that the warped metric g', of M', has the constant scalar curvature $\bar{k}$. It follows from (20) that if f is an eigenfunction, then af is also an eigenfunction for any real positive number a. Thus, there are infinitely many warped metrics all of which have constant scalar curvature $\bar{k}$. Finally, based on Proposition 4, $\bar{k}$ is also scalar curvature of the globally null manifold M and its degenerate metric g can be glued with the Riemannian metric g' as explained in the proof of Theorem 3. □

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